Introduction to Mathematics and Modeling

lecture 3
Second order differential equations

UNIVERSITY OF TWENTE.

academic year : 18-19
lecture : 3
build : February 4, 2019
slides : 29
This week

1. Section 17.1: Second-order linear differential equations
2. Section 17.2: Inhomogeneous linear equations
3. Section 17.3: Applications
A second-order linear differential equations with constant coefficients is a differential equation of the form

\[ ay'' + by' + cy = g(x) \]

where \( a, b \) and \( c \) are real constants and where \( g(x) \) is a function.

- **Second-order**: \( y'' \) appears in the equation.
- **Linear**: no nonlinear terms of \( y, y', y'' \).
- **Constant coefficients**: \( a, b, c \) are independent of \( x \).
- If \( g(x) = 0 \) then the equation is **homogeneous**.
- If \( g(x) \neq 0 \) then the equation is **inhomogeneous**. The function \( g(x) \) is called the input or forcing term.
Homogeneous second-order linear differential equations

\[ ay'' + by' + cy = 0 \]

**Superposition principle**

If \( y_1(x) \) and \( y_2(x) \) are solutions of a *homogeneous* second-order linear differential equation, then

\[ y(x) = k_1 y_1(x) + k_2 y_2(x) \]

is also a solution for all constants \( k_1 \) and \( k_2 \).

⚠️ The superposition principle does not hold for inhomogeneous differential equations.
Homogeneous second-order linear differential equations

\[ ay'' + by' + cy = 0 \]

- Two solutions \( y_1(x) \) and \( y_2(x) \) are considered to be different if one is not a scaled version of the other.

**Theorem**

If \( y_1(x) \) and \( y_2(x) \) are different solutions of a homogeneous second-order linear differential equation, then all solutions can be described by

\[ y(x) = k_1 y_1(x) + k_2 y_2(x). \]  

(\(^\ast\))

where \( k_1 \) and \( k_2 \) are arbitrary constants.

- The formula \( k_1 y_1(x) + k_2 y_2(x) \) is called the general solution.
Trial solution $y(x) = e^{\lambda x}$

\[ ay'' + by' + cy = 0 \]

- Differentiate $y(x)$:
  
  \[ \begin{align*} 
  y(x) &= \\
  y'(x) &= \\
  y''(x) &= 
  \end{align*} \]

- Substitution in the differential equation gives
Trial solution $y(x) = e^{\lambda x}$

\[ ay'' + by' + cy = 0 \]  \hspace{1cm} (1)

- If $y(x) = e^{\lambda x}$ is a solution of (1), then $a\lambda^2 + b\lambda + c = 0$.
- The zeros of equation (2) can be found with the quadratic formula:

Theorem

If $b^2 - 4ac > 0$ then the general solution of (1) is

\[ y(x) = k_1 e^{\lambda_1 x} + k_2 e^{\lambda_2 x}, \]

where

\[ \lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \]

- If $b^2 - 4ac \leq 0$, then (1) does not have two different solutions of the form $e^{\lambda x}$.  

Trial solution $y(x) = e^{\sigma x} \cos(\omega x)$ with $\omega \neq 0$

\[ ay'' + by' + cy = 0 \]

- Differentiate $y(x)$:
  
  \[ y(x) = \]
  
  \[ y'(x) = \]
  
  \[ y''(x) = \]

- Substitution in the differential equation gives

- This implies
The function \( y(x) = e^{\sigma x} \cos(\omega x) \) is a solution if and only if

\[
\begin{align*}
\quad a(\sigma^2 - \omega^2) + b\sigma + c &= 0 \\
\omega(2a\sigma + b) &= 0
\end{align*}
\]

Since \( \omega \neq 0 \) we have \( \sigma = -b/2a \), but then

\[
\omega^2 = \frac{-b^2 + 4ac}{4a^2} \quad \Rightarrow \quad \omega = \pm \frac{\sqrt{-b^2 - 4ac}}{2a}
\]

This precisely works when \( b^2 - 4ac < 0 \)!

Notice that

\[
\lambda = \sigma \pm \omega i = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

are the (complex) roots of the equation \( a\lambda^2 + b\lambda + c = 0 \).
The function \( y(x) = e^{\sigma x} \sin(\omega x) \) is a solution of \( ay'' + by' + cy = 0 \) if and only if
\[
\begin{cases}
    a(\sigma^2 - \omega^2) + b\sigma + c &= 0 \\
    \omega(2a\sigma + b) &= 0
\end{cases}
\]
These are the same conditions as for \( y(x) = e^{\sigma x} \cos(\omega x) \):

If \( y(x) = e^{\sigma x} \sin(\omega x) \) is a solution then \( y(x) = e^{\sigma x} \cos(\omega x) \) is a solution, and vice versa.

**Theorem**

If \( b^2 - 4ac < 0 \) then the general solution of \( ay'' + by' + cy = 0 \) is
\[
y(x) = e^{\sigma x} \left[ k_1 \cos(\omega x) + k_2 \sin(\omega x) \right],
\]
where
\[
\sigma = -\frac{b}{2a} \quad \text{and} \quad \omega = \frac{\sqrt{-b^2 + 4ac}}{2a}.
\]
Definition

Let \( ay'' + by' + cy = 0 \) be a homogeneous linear differential equation. The quadratic equation

\[
a\lambda^2 + b\lambda + c = 0.
\]

is called the auxiliary equation or characteristic equation.

- The roots of the auxiliary equation are determined by the quadratic formula:
  \[
  \lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.
  \]
- The number \( b^2 - 4ac \) is called the discriminant.
- There are three possibilities:
  - **Case 1**: \( b^2 - 4ac > 0 \),
  - **Case 2**: \( b^2 - 4ac < 0 \),
  - **Case 3**: \( b^2 - 4ac = 0 \).
Solution method for Case 1 (positive discriminant)

\[ a\lambda^2 + b\lambda + c = 0 \quad \Rightarrow \quad \lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

**Case 1:** \( b^2 - 4ac > 0 \)

\[ \lambda_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad \lambda_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \]

General solution:

\[
\begin{align*}
y_1(x) &= e^{\lambda_1 x} \\
y_2(x) &= e^{\lambda_2 x}
\end{align*}
\]

\[ \Rightarrow \quad y(x) = k_1 e^{\lambda_1 x} + k_2 e^{\lambda_2 x} \]
Solution method for Case 2 (negative discriminant)

\[ a\lambda^2 + b\lambda + c = 0 \quad \Rightarrow \quad \lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

Case 2: \[ b^2 - 4ac < 0 \]

\[ \lambda = \sigma \pm i\omega \quad \Rightarrow \quad \sigma = -\frac{b}{2a} \quad \text{and} \quad \omega = \frac{\sqrt{-b^2 + 4ac}}{2a} \]

General solution:

\[
\begin{align*}
y_1(x) &= e^{\sigma x} \sin(\omega x) \\
y_2(x) &= e^{\sigma x} \cos(\omega x)
\end{align*}
\]

\[ \Rightarrow \quad y(x) = k_1 e^{\sigma x} \sin(\omega x) + k_2 e^{\sigma x} \cos(\omega x) \]
Solution method for Case 3 (zero discriminant)

\[ a\lambda^2 + b\lambda + c = 0 \implies \lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

**Case 3:** \[ b^2 - 4ac = 0 \]

\[ \lambda = -\frac{b}{2a} \]

General solution:

\[
\begin{align*}
    y_1(x) &= e^{\lambda x} \\
    y_2(x) &= xe^{\lambda x}
\end{align*}
\]

\[ y(x) = k_1 e^{\lambda x} + k_2 xe^{\lambda x} \]
Example 1

\[ y'' - y' - 6y = 0 \]

- Solve the **characteristic equation** \((a = 1, b = -1, c = -6)\)

- Discriminant:

- The general solution is
  \[
  y(x) = \]

Example 2

\[ y'' - 4y' + 5y = 0 \]

- Solve the characteristic equation \((a = 1, b = -4, c = 5)\)

- Discriminant:

- Find \(\sigma\) and \(\omega\):

- The general solution is
Example 3

\[ y'' + 2y' + y = 0 \]

- Solve the **characteristic equation** \((a = 1, b = 2, c = 1)\)

- Discriminant:

- The general solution is
If
\[ ay'' + by' + cy = g(x), \]  
(1)
is a nonhomogeneous linear differential equation, then the **complementary equation of** (1) is
\[ ay'' + by' + cy = 0. \]  
(2)

**Theorem**

*Sec. 17.2, theorem 7*

The general solution to the nonhomogeneous equation (1) has the form
\[ y(x) = y_c(x) + y_p(x), \]
where the **complementary solution** \( y_c(x) \) is the general solution to the associated homogeneous equation (2), and \( y_p(x) \) is any solution to the nonhomogeneous equation (1).
If
\[ ay'' + by' + cy = g(x), \]  
(1)
is a nonhomogeneous linear differential equation, then the **complementary equation of** (1) is
\[ ay'' + by' + cy = 0. \]  
(2)

**Theorem**

The general solution to the nonhomogeneous equation (1) has the form
\[ y(x) = y_c(x) + y_p(x), \]
where the **complementary solution** \( y_c(x) \) is the general solution to the associated homogeneous equation (2), and \( y_p(x) \) is any solution to the nonhomogeneous equation (1).

- \( y_p(x) \) is called a **particular solution**.

⚠️ You only need one particular solution!
Finding a particular solution to (1) depends on $g(x)$ and is often very hard. There are several methods to find particular solutions.

The **method of Undetermined Coefficients** is based on trial solutions which are similar to $g(x)$ and that contain unknown constants.

For a full description, see Thomas’ Calculus section 17.2. We will study this method for the special case where $g(x) = g_0$ is a constant.

**Attempt 1**

If $g(x) = g_0$, try $y_p(x) = A$, where $A$ is a constant.
Example

Find the general solution to $y'' - 3y' + 2y = 6$.

- The solution to the complementary equation $y'' - 3y' + 2y = 0$ is $y_h(x) = k_1 e^x + k_2 e^{2x}$.
- Try $y_p(x) = A$. 

Example

Find the general solution to $y'' - 3y' = 6$.

- The solution to the complementary equation $y'' - 3y' = 0$ is $y_h(x) = k_1 e^{3x} + k_2$.
- Try $y_p(x) = A$. 
Definition

A second order initial or boundary value problem is a second order differential equation accompanied by two conditions.

General form of an initial value problem:

\[
\begin{align*}
ay'' + by' + cy &= g(x), \\
y(x_0) &= y_0, \\
y'(x_0) &= y_1.
\end{align*}
\] (1)

The conditions \(y(x_0) = y_0\) and \(y'(x_0) = y_1\) are called initial conditions.

Theorem

There is only one function \(y(x)\) that satisfies initial value problem (1).
Example 4

\[
\begin{align*}
y'' - 2y' + y &= 0, \\
y(0) &= 1, \\
y'(0) &= -1.
\end{align*}
\]
Example 4, continued

\[
\begin{cases}
y'' - 2y' + y = 0, \\
y(0) = 1, \\
y'(0) = -1.
\end{cases}
\Rightarrow
y(x) = e^x + k_2 x e^x
\]
Newton’s second law:

Mass times acceleration equals net force.

Newton’s second law yields the following differential equation:

$$mx''(t) + bx'(t) + kx(t) = F(t)$$
If \( b = 0 \) and \( F(t) = 0 \) then

\[ mx''(t) + kx(t) = 0. \]

The characteristic equation is

\[ m\lambda^2 + k = 0 \quad \Rightarrow \quad \lambda = \pm \sqrt{-\frac{k}{m}} \]

Because \( k, m > 0 \) the general solution is

\[ x(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t), \quad \text{where} \quad \omega = \sqrt{\frac{k}{m}}. \]

The undriven MSD-system without damping behaves as an oscillator with frequency \( \omega = \sqrt{k/m} \).
If \( F(t) = 0 \) then

\[
mx''(t) + bx'(t) + kx(t) = 0.
\]

The characteristic equation is

\[
m\lambda^2 + b\lambda + k = 0. \quad \implies \quad \lambda = -\frac{b}{2m} \pm \frac{1}{2m} \sqrt{b^2 - 4mk}
\]

A system is called **underdamped** if \( b^2 - 4mk < 0 \). In that case:

\[
x(t) = e^{-bt/2m} [c_1 \cos(\omega t) + c_2 \sin(\omega t)]
\]

where

\[
\omega = \frac{1}{2m} \sqrt{4mk - b^2}.
\]
With $F(t) = 0$ the differential equation becomes

$$mx''(t) + bx'(t) + kx(t) = 0.$$ 

The characteristic equation is

$$m\lambda^2 + b\lambda + k = 0 \implies \lambda = -\frac{b}{2m} \pm \frac{1}{2m}\sqrt{b^2 - 4mk}$$

A system is called **critically damped** if $b^2 - 4mk = 0$. In that case:

$$x(t) = c_1 e^{-bt/2m} + c_2 t e^{-bt/2m}.$$
With $F(t) = 0$ the differential equation becomes

$$mx''(t) + bx'(t) + kx(t) = 0.$$  

The characteristic equation is

$$m\lambda^2 + b\lambda + k = 0 \implies \lambda = -\frac{b}{2m} \pm \frac{1}{2m} \sqrt{b^2 - 4mk}.$$  

A system is called **overdamped** if $b^2 - 4mk > 0$. In that case:

$$x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}.$$  

Geogebra applet: [damped spring system](#)
Assignment: IMM2 - Tutorial 3.4